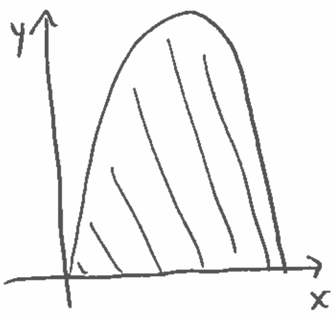
1. (10 pts)

(a) Determine the absolute area between the x-axis and $f(x) = \sin(x)\cos(x)$, from $0 \le x \le \pi/2$.

(b) Find the absolute area that is bounded by $y = \sin^2(x)$ and $y = 1$, where $\pi/2 \le x \le 3\pi/2$.

(a)

/4



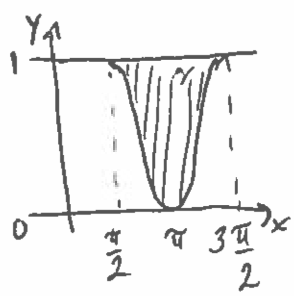
$$A = \int_0^{\pi/2} \sin x \cos x \, dx = \int_0^{\pi/2} \frac{1}{2}\sin(2x)\, dx$$

$$= -\frac{1}{4}\cos(2x)\Big|_0^{\pi/2} = -\frac{1}{4}\left(\cos(\pi) - \cos(0)\right)$$

$$= -\frac{1}{4}(-2) = \frac{1}{2}$$

(b)

/6



$$y = \sin^2(x), \; y = 1 \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

$$\sin^2(x) = 1 \Rightarrow x = \pi/2, \, 3\pi/2$$

$$A = \int_{\pi/2}^{3\pi/2} (1 - \sin^2(x))\, dx = \int_{\pi/2}^{3\pi/2} \cos^2 x \, dx$$

$$= \left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_{\pi/2}^{3\pi/2} = \frac{1}{2}\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin(3\pi) - \left[\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right]$$

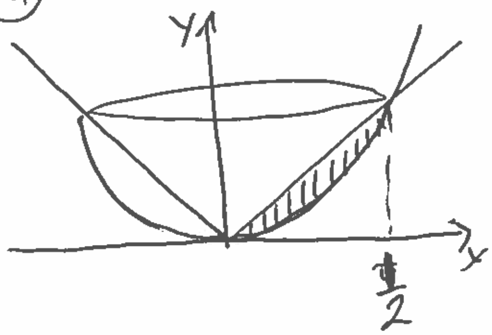
$$= \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$$

2. (10 pts)

(a) The region bounded by the curves $y = x^2$ and $x = 2y$ is rotated about the y-axis. Find the volume of the resulting solid.

(b) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point (1,1) to the point (2,1/2).

(a)



$$y = 4y^2 \Rightarrow 4y^2 - y = 0$$

$$\Rightarrow y = 0, \tfrac{1}{4}$$

$$x = 0, \tfrac{1}{2}$$

7

$$V = \int_0^{1/4} \pi\left((\sqrt{y})^2 - (2y)^2\right) dy = \pi\int_0^{1/4}(y - 4y^2)\,dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{4}{3}y^3\right)\Big|_0^{1/4} = \pi\left(\frac{1}{2}\left(\frac{1}{16}\right) - \frac{4}{3}\left(\frac{1}{64}\right)\right) = \frac{\pi}{96}$$

(b) $y = 1/x \qquad dy/dx = -1/x^2$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\,dx = \int_1^2 \frac{\sqrt{x^4+1}}{x^2}\,dx$$

3

3. (10 pts)

Evaluate the definite integrals:

(a) $\displaystyle\int_0^{\pi/4} \frac{x\sin(x)}{\cos^3(x)}dx$ (b) $\displaystyle\int_2^3 \frac{4x}{x^3 - x^2 - x + 1}dx$

(a) $\displaystyle\int_0^{\pi/4} \frac{x\sin(x)}{\cos^3(x)}dx = \int_0^{\pi/4} x\tan(x)\sec^2(x)\,dx$ let $u = x$, $dv = \sec^2 x \tan x\,dx$; $du = dx$, $v = \frac{1}{2}\sec^2 x$

$$= \frac{x\sec^2(x)}{2}\Big|_0^{\pi/4} - \int_0^{\pi/4} \frac{1}{2}\sec^2(x)\,dx$$

$$= \left(\frac{x\sec^2(x)}{2} - \frac{1}{2}\tan(x)\right)\Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$$

4

(b) $\displaystyle\int_2^3 \frac{4x}{x^3-x^2-x+1}dx = \int_2^3 \frac{4x}{(x-1)^2(x+1)}dx = \int_2^3\left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right)dx$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$
$$= A(x^2-1) + B(x+1) + C(x^2-2x+1)$$

6

$A + C = 0$ $B - 2C = 4$ $-A + B + C = 0$

$\Rightarrow A = 1,\ B = 2,\ C = -1$

$$= \int_2^3\left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right)dx = \left(\ln(x) - \frac{2}{x-1} - \ln(x+1)\right)\Big|_2^3$$

$$= \ln(2) - 1 - \ln(4) - (\ln(1) - 2 - \ln(3))$$
$$= 1 + \ln(3) + \ln(1/2) = 1 + \ln(3) - \ln(2)$$

4. (10 pts)

(a) Show that $y = e^{-\int P(x)dx} \int \left(Q(x)e^{\int P(x)dx}\right) dx$ is the general solution to the 1st-order differential equation $\frac{dy}{dx} + P(x)y = Q(x)$.

(b) Use the result from part (a) to solve the initial value problem:

$$x\frac{dy}{dx} + 2y = xe^{-x}, \quad y(1) = -1.$$

(a) let $I(x) = e^{\int P(x)dx}$, $I'(x) = P(x)e^{\int P(x)dx}$

multiply both sides by $I(x)$

$$I(x)\frac{dy}{dx} + P(x)I(x)y = I(x)Q(x)$$

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

$$\therefore y = e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$$

5

(b) $P(x) = \frac{2}{x}$ $\quad I(x) = e^{\int 2/x\, dx} = x^2$

5

$$y' + \frac{2}{x}y = e^{-x}$$

$$x^2y' + 2xy = x^2e^{-x}$$

$$x^2y = \int x^2e^{-x}dx$$

$$y = \frac{1}{x^2}\int x^2e^{-x}dx$$

$$= \frac{-1}{x^2}(2+2x+x^2)e^{-x} + \frac{C}{x^2}$$

$$y(1) = -1 = -5e^{-1} + C \quad \therefore C = 5e^{-1} - 1$$

$$\therefore y = \frac{-e^{-x}(x^2+2x+2)}{x^2} + \frac{5e^{-1}-1}{x^2}$$

5. (10 pts)

(a) Show that, in a Taylor Series expansion of $f(x)$ about the point $x = a$, the coefficients in the expansion

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

are given by $c_n = f^{(n)}(a)/n!$ where (n) indicates the nth derivative.

(b) USE part (a) to derive the Taylor/MacLaurin series for $f(x) = \dfrac{1}{1-x}$ about $x = 0$, and determine the radius of convergence.

(a) $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$

$f''(x) = 2c_2 + c_3 3\cdot 2(x-a) + 4\cdot 3\, c_4 (x-a)^2 + \ldots$

$f'''(x) = 3\cdot 2 c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + \ldots$

$\therefore f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2, \quad f'''(a) = 3\cdot 2 c_3$

$\Rightarrow f^{(n)}(a) = n!\, c_n$

$\Rightarrow c_n = \dfrac{f^{(n)}(a)}{n!}$

/7

(b) $f(x) = \dfrac{1}{1-x}$ $\quad f'(x) = \dfrac{1}{(1-x)^2}$ $\quad f''(x) = \dfrac{2}{(1-x)^3}$ $\quad \ldots$

$c_0 = \dfrac{f(0)}{0!} = 1$ $\quad c_1 = \dfrac{f'(x)\big|_{x=0}}{1!} = 1$ $\quad c_2 = \dfrac{\frac{2}{(1-x)^3}\big|_{x=0}}{2!} = 1, \ldots, \underline{\underline{c_n = 1}}$

$\therefore \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \displaystyle\sum_{n=0}^{\infty} x^n$

$|x| < 1, \quad R = 1$

/3

6. (10 pts)

(a) Evaluate $\displaystyle\int_0^3 \frac{dx}{x-2}$ if possible.

(b) Consider the probability density function $f(x) = \begin{cases} ce^{-cx} & x \geq 0 \\ 0 & x < 0 \end{cases}$,

where $c > 0$. Show that $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$, and find the mean $\mu = \displaystyle\int_{-\infty}^{\infty} xf(x)dx$.

ⓐ $\displaystyle\int_0^3 \frac{dx}{x-2} = \lim_{t\to 2^-}\int_0^t \frac{1}{x-2}\,dx + \lim_{t\to 2^+}\int_t^3 \frac{1}{x-2}\,dx$

$\displaystyle = \underbrace{\lim_{t\to 2^-} \ln|x-2|\Big|_0^t}_{A} + \underbrace{\lim_{t\to 2^+} \ln|x-2|\Big|_t^3}_{B}$

5

but $\displaystyle\lim_{t\to 2^-} \ln|x-2|\Big|_0^t = \lim_{t\to 2^-}(\ln|t-2| - \ln|-2|)$

and $\displaystyle\lim_{t\to 2^-} \ln(2-t) = -\infty$ $\therefore$ A is divergent $\Rightarrow \displaystyle\int_0^3 \frac{dx}{x-2}$ is divergent

(no need to check B)

ⓑ $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} ce^{-cx}dx = \lim_{t\to\infty}\int_0^t ce^{-cx}dx = \lim_{t\to\infty} -\frac{c}{c}e^{-cx}\Big|_0^t$

$\displaystyle = \lim_{t\to\infty} -\left[e^{-ct}-1\right] = 1$

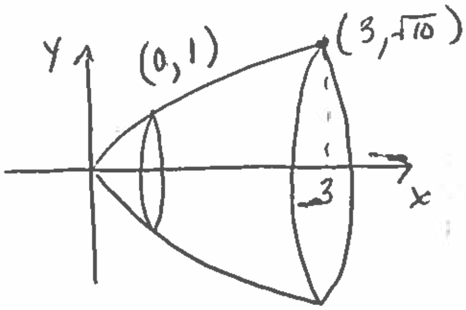
5

$\displaystyle\int_0^{\infty} cxe^{-cx}dx = \lim_{t\to\infty}\int_0^t cxe^{-cx}dx$ let $u = x$, $du = dx$, $dv = ce^{-cx}dx$, $v = -e^{-cx}$

$\displaystyle = \lim_{t\to\infty}\left(-xe^{-cx}\Big|_0^t + \int_0^t e^{-cx}dx\right) = \lim_{t\to\infty}\left(-te^{-ct} + \frac{1}{c}\left(\frac{-e^{-ct}}{\cancel{}}\right)\right) = \frac{1}{c}$

7. (10 pts)

A group of engineers is building a satellite dish whose shape will be formed by rotating $y = \sqrt{3x+1}$ about the x-axis. Find the surface area of the dish, if it is to be 3 units deep (i.e., $0 \leq x \leq 3$).



$x=0,\ y=1$

$x=3,\ y=\sqrt{10}$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x+1}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4(3x+1)}$$

$$S = \int_0^3 2\pi\sqrt{3x+1}\sqrt{1+\frac{9}{4(3x+1)}}\,dx$$

$$= \int_0^3 2\pi\sqrt{3x+1+\frac{9}{4}}\,dx = \int_0^3 \pi\sqrt{12x+13}\,dx$$

$$= \frac{\pi}{18}(-12x+13)^{3/2}\Big|_0^3 = \frac{\pi}{18}\left(49^{3/2} - 13^{3/2}\right)\text{units}^2$$

/10

8. (10 pts)

(a) For each of the following series, determine whether it converges absolutely, coverges conditionally, of diverges. Be sure to justify your answer. (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)^{3/2}}$ (ii) $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$

(b) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n^{0.9}}$.

(c) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

(a)(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)^{3/2}}$ Alternating series test $b_n = \frac{1}{(n+2)^{3/2}}$ 2

$b_{n+1} \leq b_n$ ✓ $\displaystyle\lim_{n\to\infty} \frac{1}{(n+2)^{3/2}} = 0$ ∴ convergent

(ii) $\displaystyle\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1 \neq 0$ 2

∴ $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$ diverges by the Divergence test

(b) $\displaystyle\left|\frac{2^{n+1}}{2^n}\frac{x^{n+1}}{x^n}\frac{(n+1)^{0.9}}{n^{0.9}}\right| = \left|2x\left(1+\frac{1}{n}\right)^{0.9}\right| \to 2|x|$ as $n\to\infty$ 4

$2|x|<1 \Rightarrow |x| < 1/2$ ∴ $R = 1/2$

Endpts

$x = 1/2$, diverges ∴ $[-1/2, 1/2)$ or $-1/2 \leq x < 1/2$

$x = -1/2$, converges

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \underbrace{1+1}_{2}\underbrace{-\frac{1}{2}+\frac{1}{2}}_{0}\underbrace{-\frac{1}{4}+\frac{1}{4}}_{0}\underbrace{-\frac{1}{8}+\frac{1}{8}}_{0}\underbrace{-\frac{1}{16}+\dots}_{0} \dots 0 \; 0$

∴ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

2

9. (10 pts)
A curve C is defined by the parametric curves $x = t + 1$ and $y = t^2 - 2$.
(a) Sketch the curve.
(b) Set up an integral to determine the arc length of the curve C between $0 \leq t \leq 2$. DO NOT EVALUATE.
(c) Show that C has a horizontal tangent line at the point (1,-2).

a) It may be easiest to eliminate t and write

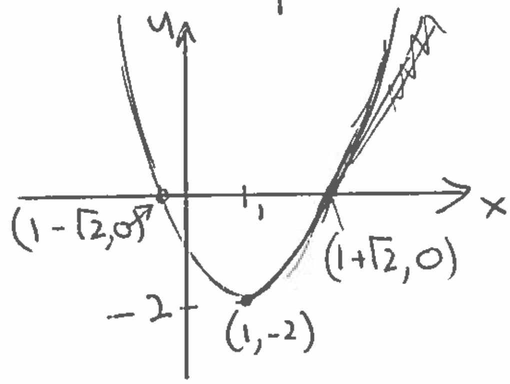
$(y+2) = (x-1)^2$ which is a parabola

$\overline{4}$

Note that when $y=0$

$(x-1)^2 = 2$

$x = 1 \pm \sqrt{2}$



b) The arc length can be written as

$\overline{4}$

$$\int_{t=0}^{t=2} ds \quad \text{where} \quad ds^2 = dx^2 + dy^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt^2$$
$$= \left[1 + (2t)^2\right] dt^2$$

So

$$\text{Arc length} = \int_0^2 \sqrt{1+4t^2}\, dt$$

c) The point $(1,-2)$ is the 'nose' of the parabola, with $t=0$.

The slope in Cartesian coordinates

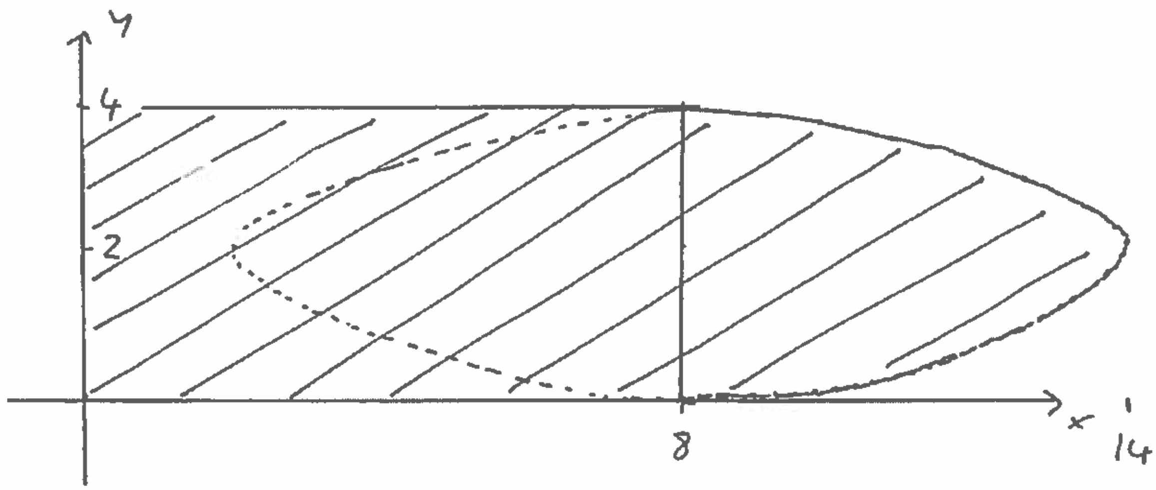
$\overline{2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1} = 0 \text{ when } t=0$$

10. (10 pts)
Leila needs to find a location for a radio repeater that will monitor the signals from a pack of radio-collared wolves. In order to get the strongest signal, she wants to locate the repeater at the centroid of the range of the wolf pack. The valley in which the wolves live encompasses roughly the union of the rectangle $0 \le x \le 8$, $0 \le y \le 4$ and part of the ellipse

$$\frac{(x-8)^2}{36} + \frac{(y-2)^2}{4} = 1$$

with $x > 8$, and all distances are in km. Where should Leila place the repeater?



Denote the centroid by $C = (\bar{x}, \bar{y})$. Because of symmetry, $\bar{y} = 2$. $\bar{x}$ is given by the formula:

$$\bar{x} = \frac{1}{A} \cdot \int_a^b x \cdot (f(x) - g(x))\, dx$$

where A is the region's area, $a = 0$, $b = 14$ and $f(\cdot)$ and $g(\cdot)$ are obtained by resolving for y in the ellipse's ~~ff~~ formula (for $x \ge 8$):

10) cont'd a)

$$y = 2 \pm \sqrt{4 - \frac{(x-8)^2}{9}}$$

$$f(x) = \begin{cases} 4 & \text{for } 0 \le x \le 8 \\ 2 + \sqrt{4 - \frac{(x-8)^2}{9}} & \text{for } 8 \le x \le 14 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x \le 8 \\ 2 - \sqrt{4 - \frac{(x-8)^2}{9}} & \text{for } 8 \le x \le 14 \end{cases}$$

The region's area is $A = 8 \cdot 4 + \frac{1}{2} \cdot A_E$ where A_E, the area of the whole ellipse, is given by $A_E = \pi \cdot \sqrt{36} \cdot \sqrt{4} = 12\pi$.$^{(*)}$ We obtain for $\bar{x}$:

$$\bar{x} = \frac{1}{A} \cdot \left[\int_0^8 x \cdot 4\,dx + \int_8^{14} x \cdot \left(2 + \sqrt{4 - \frac{(x-8)^2}{9}} - \left(2 - \sqrt{4 - \frac{(x-8)^2}{9}}\right)\right) dx \right]$$

$$= \frac{1}{A} \cdot \left[\int_0^8 4x\,dx + \int_8^{14} x \cdot 2 \cdot \sqrt{4 - \frac{(x-8)^2}{9}}\,dx \right]$$

(*) A_E can be obtained from the integral

$\frac{1}{4} \cdot A_E = \int_8^{14} \left(2 + \sqrt{4 - \frac{(x-8)^2}{9}}\right) dx = \frac{1}{3} \cdot \int_0^6 \sqrt{36 - x^2}\,dx$ using the trig. substitution $x = 6 \cdot \sin\Theta$.

10) cont'd b)

$$= \frac{1}{A} \cdot \left([2x^2]_0^8 + \int_8^{14} \frac{2}{3} x \cdot \sqrt{36-(x-8)^2}\, dx \right]$$

$$= \frac{1}{A} \cdot \left[128 + \frac{1}{3} \cdot \int_0^6 2y \cdot \sqrt{36-y^2}\, dy + \frac{16}{3} \cdot \int_0^6 \sqrt{36-y^2}\, dy \right]$$

(substituting $y = x - 8$)

For the last integral : $\int_0^6 \sqrt{36-y^2}\, dy = \frac{1}{4} \cdot \pi 6^2 = 9\pi$

For the middle term, substitute $z = 36 - y^2$

$\Rightarrow \frac{dz}{dy} = -2y$

$$\Rightarrow \int_0^6 2y \cdot \sqrt{36-y^2}\, dy = \int_0^{36} \sqrt{z}\, dz = \left[\frac{2}{3} z^{3/2} \right]_0^{36}$$

$$= 144 .$$

$$\Rightarrow \bar{x} = \frac{128 + \frac{1}{3} \cdot 144 + \frac{16}{3} \cdot 9\pi}{32 + 6 \cdot \pi} = \frac{176 + 48\pi}{32 + 6\pi}$$

$$= \underline{\underline{6.42\bar{6}7}}$$

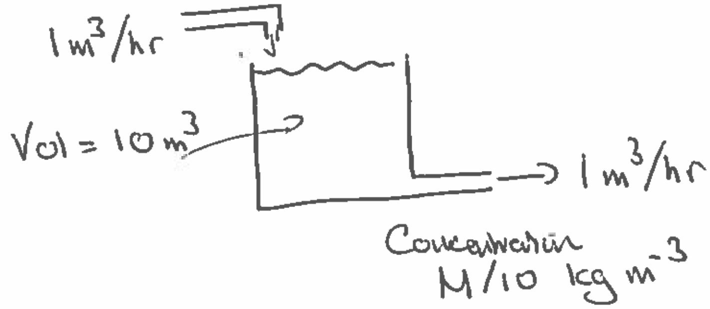
$\Rightarrow \underline{\underline{C = (6.4267, 2)}}.$

P.T.

11. (10 pts)

A 100 kg bag of salt is erroneously dropped into a fresh water tank of volume 10 m^3. The tank is assumed to be well mixed and fresh water (with zero salt concentration) is added at a steady rate of 1 m^3/hour with tank water flowing out at the same rate. Water is drinkable at salt concentration of about 0.5 g/kg. How long will it take for water in the tank to reach that level? Assume a water density of 1000 kg /m^3.

Let the mass of salt in the tank at time t be $M(t)$



We can set up the equation for the rate of change of $M(t)$ as

$$\frac{dM}{dt} = -\frac{M}{10} \quad kg/hr$$

Solution $\quad M = M_0 e^{-t/10}$ (separable or linear equation)

Note $M_0 = 100$ kg. [Initial concentration $10 kg/m^3$]

A concentration of $0.5 g/kg = 0.5 g \times 1000/m^3 = 0.5\ kg\,m^{-3}$

So we need to find t when $M = 5$ kg $(0.5 \times 10$ kg$)$

So $\quad e^{-t/10} = \frac{5}{100} = \frac{1}{20}$ $\therefore$ $\underline{\underline{t = 10 \ln 20 \text{ hours}}}$

10

PF.

If $M>P$, then increase
$M<P$, then decrease

1

12. (10 pts)

The logistic equation $\frac{dP}{dt} = kP\frac{(M-P)}{M}$ can be used to model the spread of an incurable, communicable disease in a population with P infected members out of a total population M.

(a) Briefly explain the rationale behind this model (one or two lines).

(b) If $k = 0.1$ day^{-1}, and $P/M = 0.1$ on day 0, how long will it take before half of the population is affected? (You may leave your answer in the form AlnB.)

a) The rationale is that the rate of new infections (dP/dt) would be proportional to the number of interactions between infected and uninfected members of the population which would be proportional to $P(M-P)$

infected uninfected

1

b) The logistic equation is separable

$$k\,dt = \frac{M\,dP}{P(M-P)} = \left(\frac{1}{P} + \frac{1}{M-P}\right)dP$$

Integrating both the sides

8

$$kt = \ln\frac{P}{M-P} + C$$

When $t=0$, $P/M = 0.1$ $\Rightarrow$ $C = \ln 9$

So $kt = \ln\frac{9P}{M-P}$; $\frac{9P}{M-P} = e^{kt}$

$$\Rightarrow P(9+e^{kt}) = Me^{kt}$$

$$\frac{P}{M} = \frac{1}{1+9e^{-kt}} \quad ; \quad 9e^{-kt} = \frac{M}{P} - 1$$

When half the population is infected $\frac{M}{P} = 2$ so

$9e^{-kt} = 1 \quad e^{kt} = 9 \quad t = \frac{1}{k}\ln 9 = \underline{10\ln 9 \text{ days}}$