FINAL EXAM SOLUTIONS

1(a)

$$-1 \leq \sin\left(\frac{2016}{x}\right) \leq 1 \quad \forall x > 0$$

$$\Rightarrow \quad e^{-1} \leq e^{\sin\left(\frac{2016}{x}\right)} \leq e^{1}$$

$$\Rightarrow \quad \underbrace{\sqrt{x}\, e^{-1}}_{\downarrow \atop 0} \leq \sqrt{x}\, e^{\sin\left(\frac{2016}{x}\right)} \leq \underbrace{\sqrt{x}\, e^{1}}_{\downarrow \atop 0}$$

The Squeeze Theorem $\Rightarrow \lim\limits_{x \to 0^+} \sqrt{x}\, e^{\sin\left(\frac{2016}{x}\right)} = 0$

(1b) $\left.\begin{array}{l} x \to 0 \\ x < 0 \end{array}\right\} \Rightarrow |x| = -x$

$$\Rightarrow \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{|x|}\right) = \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{-x}\right)$$

$$= \lim_{x \to 0^-} 0 = 0 \;//$$

(1c) $$\lim_{x \to \infty} \frac{\ln\sqrt{x}}{x^2} \underset{\text{L'H rule}}{=} \lim_{x \to \infty} \frac{(\ln\sqrt{x})'}{(x^2)'} = \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x}} \cdot (\sqrt{x})'}{2x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2x}}{2x} = \lim_{x\to\infty} \frac{1}{4x^2} = 0 \quad //$$

(1d) $$\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{1+2x} - \sqrt{1-4x}\right)\left(\sqrt{1+2x} + \sqrt{1-4x}\right)}{x\left(\sqrt{1+2x} + \sqrt{1-4x}\right)}$$

$$= \lim_{x\to 0} \frac{1+2x-(1-4x)}{x\left(\sqrt{1+2x} + \sqrt{1-4x}\right)} = \lim_{x\to 0} \frac{6}{\sqrt{1+2x} + \sqrt{1-4x}}$$

$$= \frac{6}{\sqrt{1}+\sqrt{1}} = \frac{6}{2} = 3 \quad //$$

(2)

$r(t)$ = the radius at time t

$A(t)$ = the surface area of the sphere at time t.

$$d(t) = 2r(t)$$

$$A(t) = 4\pi r^2(t)$$

$$\Rightarrow A(t) = 4\pi \left(\frac{d(t)}{2}\right)^2$$

$$\Rightarrow A(t) = \pi d(t)^2 \quad \Big/ \frac{d}{dt}$$

$$A'(t) = \pi 2 d(t) d'(t)$$

$$1 = 2\pi \cdot 10 \cdot d'(t)$$

$$\Rightarrow d'(t) = \frac{1}{20\pi} \ \mathrm{cm/min}.$$

(3) $f(x) = 2016\left(\sqrt{x^2+1} - x\right)$

(1) the domain of f is $\mathbb{R}$

(2) $2016\left(\sqrt{x^2+1} - x\right) = 0$, $f(0) = 2016\left(\sqrt{0^2+1} - 0\right) = 2016$ //

$\Leftrightarrow \sqrt{x^2+1} - x = 0$

$\Leftrightarrow \sqrt{x^2+1} = x$

$\Rightarrow x^2+1 = x^2$

$1 = 0$

$\Rightarrow$ no solutions. $\Rightarrow f$ is positive $\forall x \in \mathbb{R}$ //

(3.) f has no symmetry.

(4.) <u>Asymptotes</u>

- f has no vertical asymptotes $(D(f) = \mathbb{R})$
- $\lim\limits_{x\to-\infty} 2016\left(\sqrt{x^2+1} - x\right) = 2016\left(\sqrt{(-\infty)^2+1} - (-\infty)\right)$

$= 2016(\infty + \infty) = \infty$

$\lim\limits_{x\to\infty} 2016\left(\sqrt{x^2+1} - x\right) = \lim\limits_{x\to\infty} 2016 \dfrac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}$

$= 2016 \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+1} + x} = \dfrac{2016}{\infty} = 0$ //

$\Rightarrow y = 0$ is a horizontal asymptote.

5. Intervals of Increase or Decrease

$$f'(x) = 2016\left(\sqrt{x^2+1}-x\right)' = 2016\left(\frac{2x}{2\sqrt{x^2+1}}-1\right)$$

$$= 2016\left(\frac{x}{\sqrt{x^2+1}}-1\right) = 2016\,\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

- it is clear that $x-\sqrt{x^2+1}<0 \quad \forall x\in\mathbb{R}$

$\Rightarrow f'(x)<0 \quad \forall x\in\mathbb{R} \Longrightarrow f$ is decreasing on $\mathbb{R}$.

6. local max and min values.

$f'(x)\neq 0 \quad \forall x\in\mathbb{R} \Rightarrow$ there is no critical ~~values~~ numbers of f

$\Longrightarrow f$ has no loc. max or loc. min.

7. slant asymptotes

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{2016\left(\sqrt{x^2+1}-x\right)}{x} = 2016\lim_{x\to-\infty}\left(\frac{\sqrt{x^2+1}}{x}-1\right)$$

$$= 2016\lim_{x\to-\infty}\left(\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}-1\right) = 2016\lim_{x\to-\infty}\left(-\sqrt{1+\frac{1}{x^2}}-1\right)$$

$$= -2\cdot 2016 = -4032$$

$\Rightarrow m = -4032$

$$b = \lim_{x\to-\infty}\left(f(x)-mx\right) = \lim_{x\to-\infty} 2016\left(\sqrt{x^2+1}-x+2x\right)$$

$$= \lim_{x\to-\infty} 2016\left(\sqrt{x^2+1}+x\right) = \lim_{x\to-\infty} 2016\,\frac{\left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}-x} \quad 7.$$

$$= 2016 \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}-x} = \frac{1}{\infty} = 0$$

$\Rightarrow b = 0$

$\Rightarrow y = -4032x$ is a slant asymptote when $x \to -\infty$.

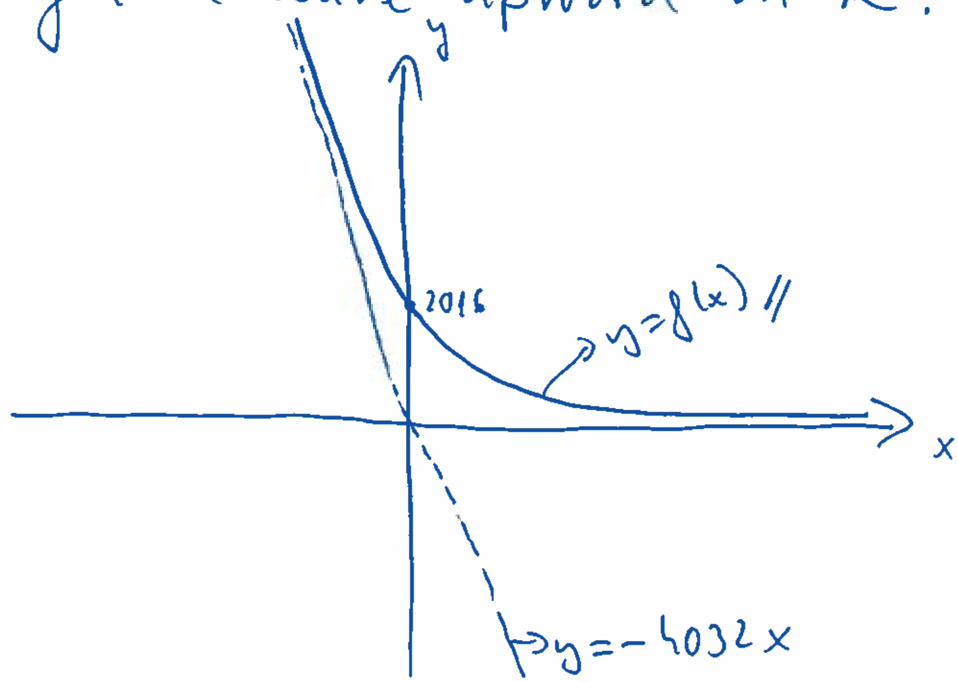
8. Concavity and Points of Inflection:

$$f''(x) = 2016\left(\frac{x}{\sqrt{x^2+1}} - 1\right)' = 2016\left(\frac{x}{\sqrt{x^2+1}}\right)'$$

$$= 2016\,\frac{\sqrt{x^2+1} - x\,\frac{1}{\cancel{2}}\,\frac{\cancel{2}x}{\sqrt{x^2+1}}}{x^2+1} = 2016\,\frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= 2016\,\frac{1}{\sqrt{x^2+1}\,(x^2+1)} > 0 \quad \forall x \in \mathbb{R}$$

$\Longrightarrow f$ is concave upward on $\mathbb{R}$.



(4a) $2x = -\cos x \Rightarrow 2x + \cos x = 0$

$$f(x) = 2x + \cos x$$

• at least one real root: $f(-1) = -2 + \cos(-1) < 0$

$f(1) = 2 + \cos 1 > 0$

~~the Intermediate Value Theorem~~

$\Rightarrow$ the Intermediate Value Theorem implies that there exists $c \in \langle -1, 1 \rangle$ such that $f(c) = 0$ //

• at most one real root.

suppose we have two real roots a and b

$f(a) = f(b) = 0$

Rolle's Theorem $\Rightarrow$ $\exists\, d \in \langle a, b \rangle$ such that $f'(d) = 0$

$f'(x) = 2 + \sin x \neq 0$ ↯↯

$\Rightarrow$ this gives a contradiction $\Rightarrow$ at most one real root //.

(4b) ~~The Mean Value Theorem $\Rightarrow$~~

↓ see page (12)

~~$\frac{f(2) - f(0)}{2 - 0} = f'(c)$, $c \in \langle 0, 2 \rangle$~~

~~$\frac{\pi + 1}{2} = f'(c) \Rightarrow f'(c) = \frac{\pi}{2}$~~

~~• This gives a contradiction because $f'(x) < \frac{1}{2}$ $\forall x \in \mathbb{R}$~~

~~$\Rightarrow$ there is no f with that property.~~

5a) $\left(\left(x+\frac{1}{x}\right)^{2016}\right)' = 2016\left(x+\frac{1}{x}\right)^{2015}\left(x+\frac{1}{x}\right)' = 2016\left(x+\frac{1}{x}\right)^{2015}\left(1-\frac{1}{x^2}\right)$ //

5b) $\left(\tan^{-1}\left(\frac{x+1}{x-1}\right)\right)' = \frac{1}{1+\left(\frac{x+1}{x-1}\right)^2}\cdot\left(\frac{x+1}{x-1}\right)' =$

$= \frac{(x-1)^2}{2(x^2+1)}\cdot\frac{-2}{(x-1)^2} = -\frac{1}{x^2+1}$ //

5c) $\left(\ln^2(\ln x)\right)' = 2\ln(\ln x)\left(\ln(\ln x)\right)' = 2\ln(\ln x)\frac{1}{\ln x}\cdot\frac{1}{x}$ //

5d) $\left((2-e^x)(x-e^x)\right)' = -e^x(x-e^x)+(2-e^x)(1-e^x)$ //

6. $\cos(x+y) = 2016x - 2016y + 1 \quad / \frac{d}{dx}$

$-\sin(x+y)(1+y') = 2016 - 2016y'$

$(x,y) = (\pi,\pi) \Rightarrow -\underbrace{\sin(2\pi)}_{=0}(1+y') = 2016 - 2016y'$

$\Rightarrow y' = 1 \Rightarrow$ the tangent line

$y = 1(x-\pi)+\pi = x$

$\Rightarrow$ the normal line:

$y = -1(x-\pi)+\pi = -x+2\pi$ //

7. $f(t)$ = the amount of tritium-3 after t years.

$$f(t) = Ce^{k \cdot t}$$

- if $t=0 \Rightarrow f(0) = C \cdot e^{k \cdot 0} = C$

$$\Rightarrow C = f(0)$$

$$\Rightarrow f(t) = f(0)e^{k \cdot t}$$

- $f(2) = 0.9 f(0) \Rightarrow 0.9 f(0) = f(2) = f(0)e^{k \cdot 2}$

$$\Rightarrow 0.9 = e^{k \cdot 2}$$

$$k \cdot 2 = \ln 0.9$$

$$k = \frac{\ln 0.9}{2}$$

- the half life: $f(t) = \frac{f(0)}{2} \Rightarrow \cancel{f(0)} e^{\frac{\ln 0.9}{2} \cdot t} = \frac{\cancel{f(0)}}{2}$

$$\Rightarrow e^{\frac{\ln 0.9}{2} \cdot t} = \frac{1}{2}$$

$$\frac{\ln 0.9}{2} \cdot t = -\ln 2 \Rightarrow t = \frac{-2\ln 2}{\ln 0.9} \text{ years}$$

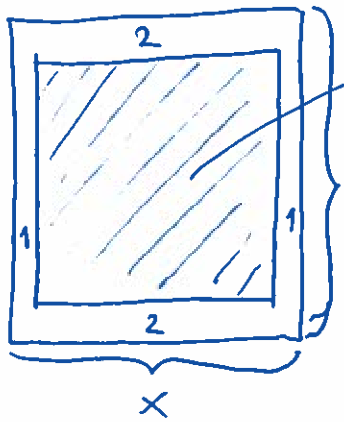
- 10% of its original amount: $f(t) = 0.1 f(0)$

$$f(0)e^{\frac{\ln 0.9}{2} \cdot t} = 0.1 \cdot f(0)$$

$$\frac{\ln 0.9}{2} \cdot t = \ln 0.1$$

$$t = \frac{2\ln 0.1}{\ln 0.9} \text{ years}$$

8.



the printed area

$240 = x \cdot y$, $x \geq 2$, $y \geq 4$

$y = \frac{240}{x}$

$A \to$ the printed area

$$A = (x-2)\cdot(y-4) = (x-2)\left(\frac{240}{x} - 4\right)$$

• find the maximum of $A(x) = (x-2)\left(\frac{240}{x} - 4\right)$ on $[2, 60]$. $A(2)=0$, $A(60)=0$

$$A'(x) = \left(\frac{240}{x} - 4\right) + (x-2)\left(-\frac{240}{x^2}\right) = \frac{240-4x}{x} - \frac{240(x-2)}{x^2}$$

$$= \frac{240x - 4x^2 - 240x + 480}{x^2} = 4\,\frac{120 - x^2}{x^2} = \frac{4}{x^2}\left(\sqrt{120} - x\right)\left(\sqrt{120} + x\right).$$

$$x = \sqrt{120} = \sqrt{4 \cdot 30} = 2\sqrt{30}$$

$A'(2\sqrt{30}) = 0$, $A'(x) > 0$, if $x \in [2, 2\sqrt{30})$, and

$A'(x) < 0$, if $x > 2\sqrt{30}$, $x \leq 60$ $\xRightarrow{\text{The First Derivative Test}}$

A has an absolute max at $x = 2\sqrt{30}$ on $[2, 60]$.

$$\Rightarrow \quad y = \frac{240}{x} = \frac{240}{2\sqrt{30}} = \frac{120}{\sqrt{30}}$$

(9a) $f(x)=x^2|x|$ is continuous on $(-\infty,\infty)$.

1.) $x<0 \Rightarrow f(x)=-x^3$ is continuous.

2.) $x>0 \Rightarrow f(x)=x^3$ is continuous.

3.) $x=0$ $\left.\begin{array}{l}\lim\limits_{x\to 0^-} x^2|x| = \lim\limits_{x\to 0^-} -x^3 = 0 \\ \lim\limits_{x\to 0^+} x^2|x| = \lim\limits_{x\to 0^+} x^3 = 0\end{array}\right\} \Rightarrow$

$\Rightarrow \lim\limits_{x\to 0} x^2|x| = 0$

$\Downarrow$

$f(0)=0 \Rightarrow \lim\limits_{x\to 0} f(x) = f(0) \Rightarrow f$ is continuous at $x=0$.

9(a) $\downarrow$ see page 13

. f is diff. on $(-\infty,\infty)$:

1.) $x<0 \Rightarrow f(x)=-x^3$ is diff.

2.) $x>0 \Rightarrow f(x)=x^3$ is diff.

3.) $x=0$ $\left.\begin{array}{l}\lim\limits_{h\to 0^-} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{-h^3}{h} = \lim\limits_{h\to 0^-} (-h^2) = 0 \\ \lim\limits_{h\to 0^+} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{h^3}{h} = \lim\limits_{h\to 0^+} h^2 = 0.\end{array}\right\} =$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = 0$ exists.//

$\Rightarrow f'(0)=0$//.

(9b) $f(x)=\dfrac{x^3-2}{x^2+2}$

$m=\lim\limits_{x\to\pm\infty} \dfrac{x^3-2}{(x^2+2)\cdot x} = \lim\limits_{x\to\pm\infty} \dfrac{x^3-2}{x^3+2x} = \lim\limits_{x\to\pm\infty} \dfrac{1-\frac{2}{x^3}}{1+\frac{2}{x^2}} = 1$//

(11.)

$$b = \lim_{x\to\pm\infty} (f(x) - mx) = \lim_{x\to\pm\infty} \left(\frac{x^3-2}{x^2+2} - x \right)$$

$$= \lim_{x\to\pm\infty} \frac{\cancel{x^3}-2-\cancel{x^3}-2x}{x^2+2} = \lim_{x\to\pm\infty} \frac{-2-2x}{x^2+2}$$

$$= \lim_{x\to\pm\infty} \frac{-\frac{2}{x^2}-\frac{2}{x}}{1+\frac{2}{x^2}} = \frac{0}{1} = 0$$

$$\boxed{y = x}$$

(5c) $$\lim_{x\to 0^+} (\tan x)^{2016x} = \lim_{x\to 0^+} e^{2016x\cdot\ln(\tan x)} = ??$$

$$0^0$$

$$\lim_{x\to 0^+} x\cdot\ln(\tan x) = \lim_{x\to 0^+} \frac{\ln(\tan x)}{\frac{1}{x}} \underset{\substack{\frac{-\infty}{\infty} \\ L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\tan x}(\tan x)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} - \frac{x^2}{\sin x\cos x}$$

$$= \lim_{x\to 0^+} - \frac{x^2}{\frac{\sin(2x)}{2}} = \lim_{x\to 0^+} -2\frac{x^2}{\sin(2x)} \underset{\substack{L'H \\ \frac{0}{0}}}{=} \lim_{x\to 0^+} -2\cdot\frac{2x}{\cos(2x)\cdot 2}$$

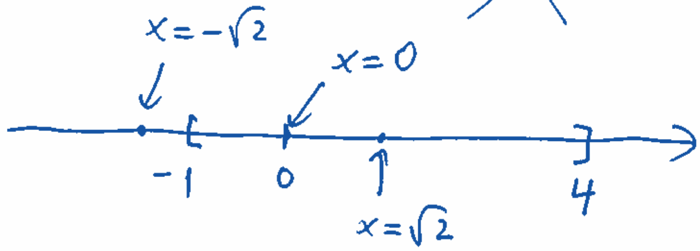
$$= \lim_{x\to 0^+} \frac{-2x}{\cos(2x)} = \frac{-2\cdot 0}{1} = 0 //$$

$$\Rightarrow \lim_{x\to 0^+} e^{2016x\ln(\tan x)} = e^{2016\cdot 0} = 1 //$$

10. $f(x) = (x^2-2)^3$ on $[-1,4]$.

$$f'(x) = 3(x^2-2)^2(x^2-2)' = 3(x^2-2)^2 \cdot 2x$$
$$= 6x(x^2-2)^2$$

$x=0$, $\quad x=\sqrt{2}$, $\quad \cancel{x=-\sqrt{2}}$



$$\left.\begin{aligned} f(0) &= (0^2-2)^3 = -8 \\ f(\sqrt{2}) &= ((\sqrt{2})^2-2)^3 = 0 \\ f(-1) &= ((-1)^2-2)^3 = (1-2)^3 = -1 \\ f(4) &= (16-2)^3 = 14^3 \end{aligned}\right\} \quad \begin{aligned} \max &= 14^3 \\ \min &= -8 \end{aligned}$$

//

(45) define $s(x) = f(x) - g(x)$

The Mean Value Theorem $\Longrightarrow$ $\dfrac{s(2016)-s(a)}{2016-a} = s'(c)$

$c \in \langle a, 2016 \rangle$.

$$\left.\begin{aligned} &\bullet\ s'(c) = f'(c) - g'(c) < 0 \\ &\bullet\ 2016 - a > 0 \end{aligned}\right\} \Longrightarrow s(2016) - s(a) < 0$$

$\bullet\ s(a) = f(a) - g(a) = 0$

$$s(2016) < 0$$
$$f(2016) - g(2016) < 0$$

//

(9a) $f(x) = |x|^x = e^{x\ln|x|}$, $x \neq 0$

$f(0) = 1$

- $x < 0 \Rightarrow |x| = -x \Rightarrow f(x) = e^{x\ln(-x)}$ is continuous on $\langle -\infty, 0)$.
- $x > 0 \Rightarrow |x| = x \Rightarrow f(x) = e^{x\ln x}$ is continuous on $\langle 0, +\infty)$.
- $x = 0 \Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^{x\ln(-x)} =$

$$= e^{\lim\limits_{x\to 0^-} x\ln(-x)} = e^{\lim\limits_{x\to 0^-} \frac{\ln(-x)}{\frac{1}{x}}} \underset{L'H}{=} e^{\lim\limits_{x\to 0^-} \frac{\frac{1}{-x}\cdot(-1)}{-\frac{1}{x^2}}}$$

$$= e^{\lim\limits_{x\to 0^-}(-x)} = e^0 = 1$$

$$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^{x\ln|x|} = \lim\limits_{x\to 0^+} e^{x\ln x} \underset{L'H}{=} e^0 = 1$$

$$\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 1$$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 1 = f(0) \Rightarrow f$ is continuous at $x = 0$

$\Rightarrow f$ is continuous on $\langle -\infty, \infty)$.