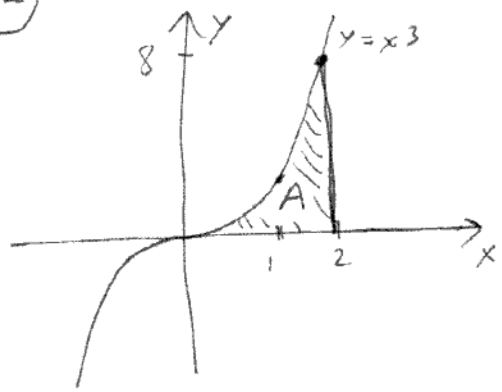
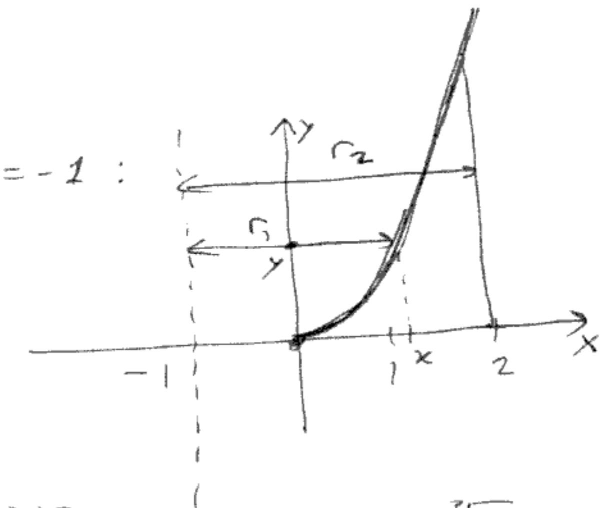
2



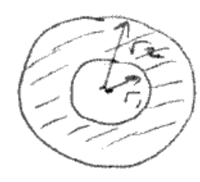
1a rotate around $x = -1$:



Note: $r_2 = 3$

since $y = x^3$ Then $x = \sqrt[3]{y}$, so $r_1 = 1 + \sqrt[3]{y}$

The cross-section is a washer



$$A(y) = \pi r_2^2 - \pi r_1^2 =$$

$$= \pi \left(9 - (1 + y^{\frac{1}{3}})^2\right)$$

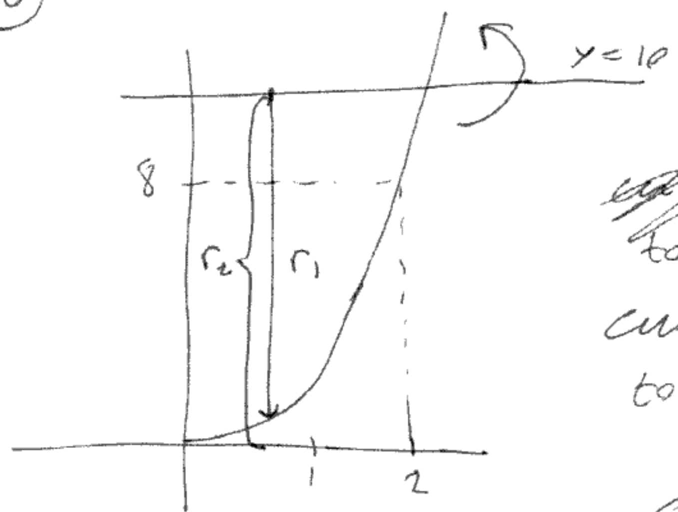
$$\text{Volume} = \int_0^8 A(y)\,dy = \int_0^8 \pi \left(9 - (1+y^{\frac{1}{3}})^2\right) dy$$

1a – Alternative solution

use the method of cylindrical shells

$$\text{Volume} = \int_0^2 2\pi (1+x)\, x^3\, dx$$

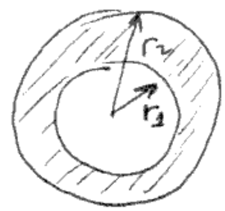
16



~~cut by plane parallel to~~

cut by planes perpendicula
to line $y = 10$

cross section is



with $r_1 = 10 - x^3$
and $r_2 = 10$

Thus $A(x) = \pi r_2^2 - \pi r_1^2 =$

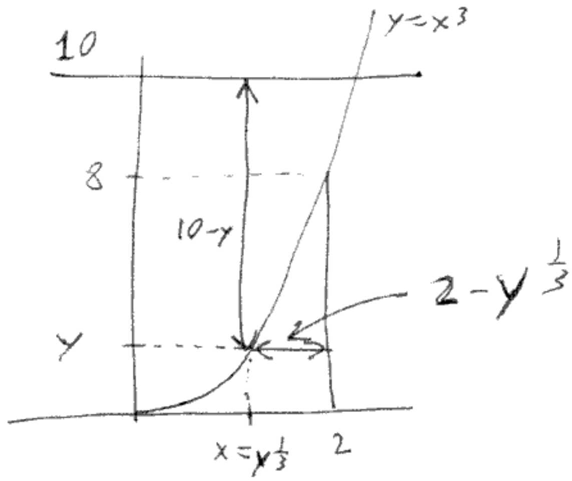
$$= \pi \left(100 - (10 - x^3)^2\right)$$

and $\text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 \pi \left(100 - (10 - x^3)^2\right) d$

16 — <u>Alternative solution</u>

use the method of cylindrical shells

$$\text{Volume} = \int_0^8 2\pi (10 - y)\left(2 - y^{\frac{1}{3}}\right) dy$$



2a $\int \tan^5(x)\cos^7(x)\,dx = \int \left(\frac{\sin(x)}{\cos(x)}\right)^5 \cos^7(x)\,dx =$

$= \int \sin^5(x)\cos^2(x)\,dx = \left\{ \begin{array}{l} u = \cos(x) \\ du = -\sin(x)\,dx \end{array} \right\} =$

$= \int \sin^4(x)\cos^2(x)\,\sin(x)\,dx = -\int (1-u^2)^2 u^2\,du$

$= -\int (1 + u^4 - 2u^2)u^2\,du =$

$= -\int (u^2 + u^6 - 2u^4)\,du = -\frac{u^3}{3} - \frac{u^7}{7} + \frac{2u^5}{5} +$

$= -\frac{1}{3}\cos^3(x) - \frac{1}{7}\cos^7(x) + \frac{2}{5}\cos^5(x) + C$

2B $\int_0^1 x\ln(1+x^2)\,dx = \left\{ \begin{array}{l} \text{integration by parts} \\ f(x) = \ln(1+x^2) \\ f'(x) = \frac{2x}{1+x^2} \\ g'(x) = x\,;\ g(x) = \frac{x^2}{2} \end{array} \right\} =$

$= \frac{x^2}{2}\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{x^3}{1+x^2}\,dx = \frac{1}{2}\ln(2) - \int_0^1 \frac{x^3}{1+x^2}\,dx$

In integral $\int_0^1 \frac{x^3}{1+x^2}\,dx$ do substitution $1+x^2 = u$

so $du = 2x\,dx$ and $\int_0^1 \frac{x^3}{1+x^2}\,dx = \int_0^1 \frac{\frac{1}{2}x^2\, 2x\,dx}{1+x^2} =$

$= \int_1^2 \frac{\frac{1}{2}(u-1)}{u}\,du = \int_1^2 \left(\frac{1}{2} - \frac{1}{2u}\right)du = \left(\frac{u}{2} - \frac{1}{2}\ln|u|\right)\Big|_1^2 =$

$= \frac{1}{2} - \frac{1}{2}\ln(2)$

Answer: $\int_0^1 x\ln(1+x^2)\,dx = \frac{1}{2}\ln(2) - \left(\frac{1}{2} - \frac{1}{2}\ln(2)\right) =$

$= \ln(2) - \frac{1}{2}$

(2B) Alternative solution:

$$\int_0^1 x \ln(1+x^2)\,dx = \left\{ \begin{array}{l} \text{substitution} \quad u = 1+x^2 \\ \qquad\qquad\qquad du = 2x\,dx \end{array} \right\} =$$

$$= \frac{1}{2}\int_1^2 \ln(u)\,du = \left\{ \begin{array}{l} \text{integration by parts} \\ f(u) = \ln(u),\ f'(u) = \frac{1}{u} \\ g'(u) = 1;\ \ g(u) = u \end{array} \right\}$$

$$= \frac{1}{2}\left[u\ln(u)\Big|_1^2 - \int_1^2 \frac{1}{u}\cdot u\cdot du \right]$$

$$= \frac{1}{2}\left[2\ln(2) - 1 \right] = \ln(2) - \frac{1}{2}$$

(2c)

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \left\{ \begin{array}{l} x = 3\sin(\theta) \\ 9-x^2 = 9\cos^2(\theta) \\ dx = 3\cos(\theta)\,d\theta \end{array} \right\} =$$

$$= \int \frac{9\sin^2(\theta)}{3\cos(\theta)}\,3\cos(\theta)\,d\theta = 9\int \sin^2(\theta)\,d\theta$$

$$= 9\int \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)$$

Note that $\theta = \sin^{-1}\left(\frac{x}{3}\right)$ and

$$\frac{1}{2}\sin(2\theta) = \sin(\theta)\cos(\theta) = \frac{x}{3}\sqrt{1-\frac{x^2}{9}}$$

Answer: $\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\sqrt{1-\frac{x^2}{9}}\right) + C$

or $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$

(2d) $I = \int \frac{y}{(y+4)(2y-1)} \, dy$

use partial fractions:

$$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$$

multiply by $(y+4)(2y-1) \Rightarrow$

$$y = A(2y-1) + B(y+4)$$

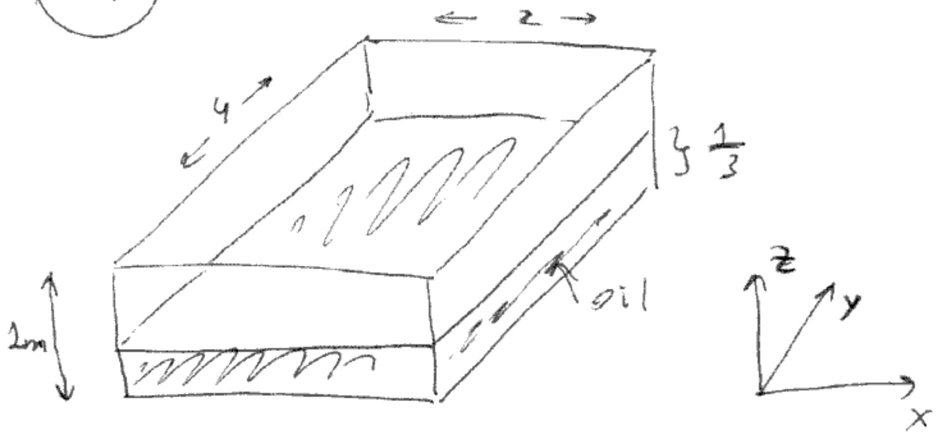
plug $y = -4 \Rightarrow -4 = A(-9)$, so $\boxed{A = \frac{4}{9}}$

plug $y = \frac{1}{2} \Rightarrow \frac{1}{2} = B\left(\frac{9}{2}\right)$ so $\boxed{B = \frac{1}{9}}$

Thus $I = \int \frac{4}{9} \frac{1}{y+4} \, dy + \int \frac{1}{9} \frac{1}{2y-1} \, dy =$

$$= \frac{4}{9} \ln|y+4| + \frac{1}{18} \ln|2y-1| + C$$

#3



consider a layer of oil at height z of thickness Δz

its weight is $4 \cdot 2 \cdot \Delta z \cdot 900 \cdot 9.8$ (N)

we need to lift it at height $(1-z)$ (m)

Work needed $= (1-z) \cdot \text{weight} =$

$$= (1-z)\ 4 \cdot 2 \cdot \Delta z \cdot 900 \cdot 9.8$$

Total work $= \int_0^{\frac{1}{3}} (1-z)\ 4 \cdot 2 \cdot 900 \cdot 9.8 \cdot dz$

$$= 7200 \cdot 9.8 \cdot \int_0^{\frac{1}{3}} (1-z)\, dz$$

$$= 7200 \cdot 9.8 \cdot \left(z - \frac{z^2}{2}\right)\Big|_0^{\frac{1}{3}} =$$

$$= 7200 \cdot 9.8 \cdot \left(\frac{1}{3} - \frac{1}{9 \cdot 2}\right) \text{ Joules}$$