1. (10 pts total)

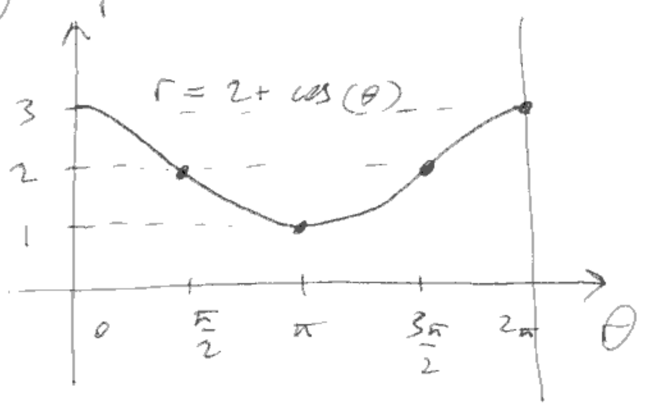
(1a) (4 points) Sketch the curve, whose equation in polar coordinates is

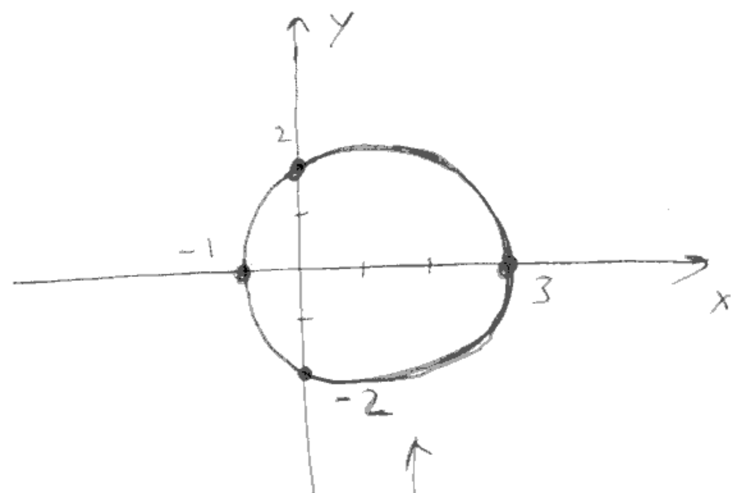
$$r = 2 + \cos(\theta), \quad 0 \le \theta \le 2\pi.$$

(1b) (3 points) Write down the integral for the area enclosed by this curve. **Do not evaluate the integral.**

(1b) (3 points) Write down the integral for the length of the part of this curve that lies in the second quadrant. The second quadrant is where $x < 0$ and $y > 0$. **Do not evaluate the integral.**

(1a)





Answer for (1a)

(1B) $\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (2 + \cos(\theta))^2 \, d\theta$

(1c) $\text{length} = \int_{\frac{\pi}{2}}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta =$

2nd quadrant $\frac{\pi}{2} \le \theta \le \pi$

$= \int_{\frac{\pi}{2}}^{\pi} \sqrt{(2 + \cos(\theta))^2 + (\sin(\theta))^2} \, d\theta$

2. (16 pts) Determine whether the series is convergent or divergent. Make sure to state which test or theorem you are using and to verify all conditions of the relevant test or theorem.

(2a) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$

(2b) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$

(2c) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

(2d) (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-\frac{1}{n}}$

(2a) Note that $1+n < 10^n$ for $n \geq 1$

Then $\dfrac{1}{\ln(1+n)} > \dfrac{1}{\ln(10^n)} = \dfrac{1}{n \ln(10)}$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln(10)}$ diverges.

By Comparison Test, the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$ also diverges.

(2B) let $a_n = (-1)^n \dfrac{3^n}{n!}$. Use Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = 3 \frac{n!}{(n+1)!} = \frac{3}{n+1} \longrightarrow 0 \text{ as } n \to +\infty.$$

By Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$ converges.

Extra space for Question 2

2c let $a_n = \dfrac{n^{2n}}{(1+2n^2)^n}$

use Root Test: $\sqrt[n]{|a_n|} = \dfrac{n^2}{1+2n^2} = \dfrac{1}{\frac{1}{n^2}+2} \to$

as $n \to +\infty$

Thus, by Root Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ converges.

2d let $a_n = (-1)^n e^{-\frac{1}{n}}$

Note that $\lim\limits_{n\to+\infty} \frac{1}{n} = 0$, so $\lim\limits_{n\to+\infty} e^{-\frac{1}{n}} = 1$

Thus $\lim\limits_{n\to+\infty} a_n \neq 0$.

By Divergence Test, the series

$\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-\frac{1}{n}}$ diverges.

3. (6 pts) Find the sum of the series $\sum_{k=1}^{\infty} \frac{2^{k+2}}{3^{k+3}}$.

$$\sum_{k=1}^{\infty} \frac{2^{k+2}}{3^{k+3}} = \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} + \cdots$$

$$= \frac{2^3}{3^4} \underbrace{\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right)}_{\substack{\text{geometric series} \\ \text{with } r = \frac{2}{3}}} \quad *$$

Since $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ for $|r| < 1$,

we have $* = \frac{2^3}{3^4} \frac{1}{1-\frac{2}{3}} = \frac{2^3}{3^4} \frac{1}{1/3} = \frac{2^3}{3^3}$

4. (8 pts) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{3^n x^n}{\sqrt{n}}$.

let $a_n = (-1)^n \frac{3^n x^n}{\sqrt{n}}$

Then $\sqrt[n]{|a_n|} = \frac{3|x|}{n^{\frac{1}{2n}}} \longrightarrow 3|x|$ as $n \to +\infty$

By Root Test, the series converges if $3|x| < 1$, ~~so that~~ $\iff |x| < \frac{1}{3}$

so the radius of convergence is $R = \frac{1}{3}$

Check the boundary points

$x = \frac{1}{3}$: $\sum_{n=1}^{\infty}(-1)^n \frac{3^n(\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Converges, By alternating series test

(Since $\frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to+\infty}\frac{1}{\sqrt{n}} = 0$)

$x = -\frac{1}{3}$: $\sum_{n=1}^{\infty}(-1)^n \frac{3^n(-\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$

Diverges (use $\sum_{n=1}^{\infty}\frac{1}{n^p}$ series with $p = \frac{1}{2}$)

Answer: $R = \frac{1}{3}$; interval $= (-\frac{1}{3}; \frac{1}{3}]$